Averkov, Gennadiy, Integer and discrete quantitative Helly numbers

I will present results obtained in a joint work with Bernardo González Merino, Matthias Schymura, Ingo Paschke and Stefan Weltge.

Let S be a discrete subset of $\mathbb{R}^n$ and define $c(S, k)$ as the smallest number with the property that if a finite family of convex sets has exactly k points of S in common, then at most $c(S, k)$ convex sets in this family already have exactly k points of S in common. For $S = \mathbb{Z}^n$, this number repeatedly appeared in different contexts as, for instance, optimization and geometry of numbers and, very recently, for general sets S, in the context of Helly and Tverberg theorems.

We give a useful description of $c(S, k)$ in terms of polytopes with vertices in S. Starting with this description, we answer several fundamental questions about $c(S, k)$. We provide the general upper bound $c(S, k) \leq \lfloor (k+1)/2 \rfloor \, (c(S, 0) - 2) + c(S, 0)$ for every discrete S. For the integer lattice $S = \mathbb{Z}^n$, employing theory of lattice polytopes and the geometry of numbers, we solve the question on the asymptotic behavior by proving the estimate $c(\mathbb{Z}^n, k) = \Theta(k^{(n-1)/(n+1)})$ for every fixed n, and we compute the exact values of $c(\mathbb{Z}^n, k)$ for $k = 0, \ldots, 4$.