Pabiniak, Milena, Okounkov bodies and toric degenerations in symplectic geometry

A toric degeneration is a construction from algebraic geometry which allows us to "degenerate" a given projective manifold M to some (symplectic) toric variety X_0, i.e. form a flat family over $\mathbb{C}$ with a generic fiber M and a special fiber X_0. As observed by Harada and Kaveh, if M is symplectic then there exists an open dense subset of M and an open dense subset of X_0 which are symplectomorphic. This allows us to study symplectic invariants of M by studying X_0 which, as a toric variety, is usually much better understood. In certain nice situations the whole M is symplectomorphic to X_0 and the degeneration provides a symplectomorphism.

To construct a toric degeneration one needs a (very ample) line bundle over M and a (nice enough) valuation on the coordinate ring $\mathbb{C}(M)$. From these data one forms a semi-group S, and an Okounkov body P, which is a rational polytope if S is finitely generated. In that case the special (toric) fiber X_0 of the degeneration will be $\mathrm{Proj}(\mathbb{C}[S])$ and its normalization will be the normal toric variety associated to the polytope P.

In this talk I will describe the construction of toric degeneration and briefly discuss two of its applications in symplectic geometry. 1. To find lower bounds for the Gromov width of M, concentrating mostly on the case when M is a coadjoint orbit of a group G (based on projects with I. Halacheva, and with X. Fang and P. Littelmann). Here we will see interaction with representation theory: for appropriately chosen line bundle and valuation, the associated Okounkov body is equal to a string polytope, i.e. a polytope obtained from a string parametrization of a crystal basis of a representation of G.
2. To study questions of the form: given two (symplectic) Bott manifolds and an isomorphism F between their integral cohomology rings, sending $[\omega_1]$ to $[\omega_2]$, is there a diffeomorphism inducing F? (Based on a project with S. Tolman.)