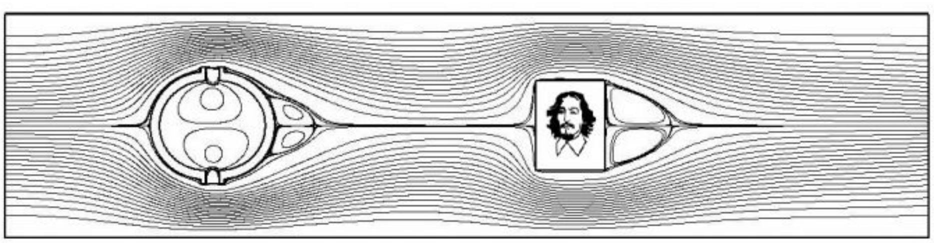
VMS-2016

11th International Workshop on Variational Multiscale and Stabilized Finite Elements

March 16 - 18, 2016

Magdeburg, Germany



Institute for Analysis and Numerics, Department of Mathematics
Otto-von-Guericke University Magdeburg

General Information

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Instructions for Speakers

Please **do not exceed 30 minutes** for the presentation of your work, such that at least **5 minutes remain for discussion.** The most comfortable way for your beamer presentation is that you **bring your talk as a pdf-file** on your USB-stick and copy it sufficiently early to the presentation laptop. As presentation software we provide Adobe Acrobat Reader DC on a Windows system. For using your own Mac laptop, we will provide a video adapter to the beamer cable. You can also use a blackboard with chalk.

Lunch

You may have lunch in the Mensa of the Otto-von-Guericke University Magdeburg. By **showing your name badge with the VMS-2016 logo** at the cashpoint you only need to pay the prize for staff members of the University. Several meal options are available at the Mensa which can be viewed in a vitrine. At lunchtime we will group and walk together to the Mensa.

Conference Dinner

The Conference Dinner will be held on **Thursday, March 17**, from **7:00 to 12:00 pm** at the **Hotel RATSWAAGE** in the center of Magdeburg. To find the room for the dinner in the hotel, please look for the sign "Workshop VMS-2016".

PROGRAM of VMS-2016

Wednesday, 16.03.2016

13:30–14:05	*Registration*	
14:05–14:15	*Opening remarks*	
14:15–14:50	**Gert Lube** (University of Göttingen) *On semirobust error estimates and efficient simulation of incompressible flows*	p. 7
14:50–15:25	**Daniel Arndt** (University of Göttingen) *Stabilized Finite Element Methods for Rotating Oberbeck-Boussinesq Flow*	p. 8
15:25–16:00	**Philipp W. Schroeder** (University of Göttingen) *Stabilised dG-FEM for thermally-coupled incompressible flows*	p. 9
16:00–16:30	*Coffee break*	
16:30–17:05	**Kristin Simon** (University of Magdeburg) *Local Projection Stabilization for Surface PDEs*	p. 10
17:05–17:40	**Steffen Basting** (University of Dortmund) *An arbitrary Lagrangian-Eulerian finite element approach for free and moving boundary problems with large deformations*	p. 11
17:40–18:15	**J. Van der Vegt** (University of Twente) *A Local Discontinuous Galerkin Method for the Navier-Stokes-Korteweg Equations*	p. 12

Thursday, 17.03.2016

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09:35–10:10	**Dmitri Kuzmin** (University of Dortmund) *Gradient-based nodal limiters for continuous finite elements*	p. 14
10:10–10:40	*Coffee break*	
10:40–11:15	**Volker John** (Weierstrass Institute Berlin) *A Review of Variational Multiscale Methods for the Simulation of Turbulent Incompressible Flows*	p. 15
11:15–11:50	**E. Delgado-Ávila** (University of Sevilla) *Reduced basis method for turbulent flows*	p. 16
11:50–12:25	**Benjamin Krank** (Technical University of Munich) *A discontinuous Galerkin solver for the simulation of incompressible turbulent flow*	p. 17
12:25–15:00	*Lunch, visiting the Cathedral of Magdeburg*	
15:00–15:35	**Ramon Codina** (Universitat Politècnica de Catalunya, Barcelona) *A stabilized finite element method for the two-field and three-field Stokes eigenvalue problems*	p. 18
15:35–16:10	**Stefan Turek** (University of Dortmund) *Numercial studies regarding accuracy, robustness and solver efficiency of special "divergence-free" finite element discretizations*	p. 19
16:10–16:40	*Coffee break*	
16:40–17:15	**Alexander Linke** (Weierstrass Institute Berlin) *Towards pressure-robust mixed methods for the incompressible Navier–Stokes equations*	p. 20
17:15–17:50	**Christian Merdon** (Weierstrass Institute Berlin) *Pressure-robust 1st and 2nd-order finite element methods for Navier–Stokes discretisations*	p. 21
19:00	*Conference Dinner* (Hotel Ratswaage)	

Friday, 18.03.2016

09:00–09:35	**Winnifried Wollner** (Technical University of Darmstadt) *Optimal L^2 Error for a Modified Crouzeix-Raviart Stokes Element*	p. 22
09:35–10:10	**Malte Braack** (University of Kiel) *An a priori error estimate of FE schemes for Navier-Stokes solutions with outflow condition*	p. 23
10:10–10:45	**Piotr Skrzypacz** (University of Astana) *On the construction of L_2 orthogonal elements of arbitrary order for Local Projection Stabilization*	p. 24
10:45–11:15	*Coffee break*	
11:15–11:50	**Gunar Matthies** (Technical University of Dresden) *Higher order variational time discretisations for the Oseen equations*	p. 25
11:50–12:25	**Friedhelm Schieweck** (University of Magdeburg) *Error analysis for a post-processed dG-solution in time of the transient Stokes problem*	p. 26
12:25	Closing remarks	
12:30	Lunch / end of the workshop	

Book of Abstracts
(in order of presentation)

On semirobust error estimates and efficient simulation of incompressible flows

G. Lube*

* Georg-August University Göttingen, Institut for Numerical and Applied Mathematics, Germany

lube@math.uni-goettingen.de

We consider some aspects of the numerical simulation of the time-dependent incompressible Navier-Stokes problem (for simplicity for no-slip conditions) using inf-sup stable FE spaces for velocity and pressure. Under the assumption $\mathbf{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega))]^n$, we derive *semirobust* semidiscrete error estimates for the kinetic and dissipation energy, i.e. the right-hand side bounds depend only implicitely on the Reynolds number through Sobolev norms of the solution [1]. Such result can be found for equal-order approximation in [2]. It turns out that grad-div stabilization is essential. We briefly discuss some open questions, in particular, whether local projection stabilization is sufficient as implicit LES model for turbulent flows. Pressure-correction schemes with BDF(2) time discretization are appropriate for the efficient parallel solution. It is shown in [1] that semirobust error estimates are valid for such schemes too. Finally, we address the efficient resolution of boundary layer flows. The enrichment approach based on the law of the wall in [4] has the potential to break the complexity problem for high Reynolds number. We conjecture that this approach can be extended to boundary layer flows with adverse pressure gradient using the generalized wall law in [3].

References

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Stabilized Finite Element Methods for Rotating Oberbeck-Boussinesq Flow

D. Arndt*

* Georg-August-Universität Göttingen, Institute for Numerical and Applied Mathematics, Germany

d.arndt@math.uni-goettingen.de

In this talk, we discuss a proper choice of stabilization for laminar and turbulent non-isothermal incompressible flow in a possibly rotating frame of reference. In particular, we consider grad-div and LPS stabilization for the momentum equation and LPS stabilization for the convective term in the Fourier equation for the temperature in the Oberbeck-Boussinesq model. Analytically, quasi-optimal and semirobust error estimates for a splitting scheme based on BDF2 are derived. The scheme is tested on Rayleigh-Bénard convection in a parameter regime from laminar to transient flow both for a non-rotating and a fast rotating boundary. The results obtained in [1, 3] are compared to DNS calculations in [2, 4, 5].

References

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Stabilised dG-FEM for thermally-coupled incompressible flows

Philipp W. Schroeder*
*Institute for Numerical and Applied Mathematics, University of Göttingen, Germany.
philippwilliam@aol.com

Discontinuous Galerkin Finite Element Methods (dG-FEM) are nowadays widely used for the simulation of modern fluid dynamics phenomena [1]. Especially the natural treatment of convection-dominated problems and the superior mass conservation properties are appealing for incompressible flows [2]. In this talk we want to investigate different stabilisation mechanisms which can be considered in the context of dG-FEM — namely pressure jump [3] and grad-div stabilisation [4]. We compare various methods by their performance on the differentially heated cavity (DHC) problem for small to high Rayleigh numbers in the laminar regime on non-adaptive meshes [5, 6]. It turns out that stabilised dG-FEM yield meaningful results even on obviously under-resolved meshes. Furthermore, we observe that divergence stabilisation complements clearly better with non-conforming methods than with comparable conforming FEM.

References

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Local Projection Stabilization for Surface PDEs

K. Simon* and L. Tobiska
Otto-von-Guericke University, Institute for Analysis and Numerics, Germany
Kristin.Simon@ovgu.de

Partial differential equations on surfaces are an area of recent research. Linear and higher order finite elements for the Laplace-Beltrami equation on surfaces are introduced in [2]. An extension to general elliptic problems on moving surfaces and a good overview over finite element methods on surfaces can be find in [3].

We consider the surface diffusion-convection-reaction equation. As known finite element methods for diffusion-convection equations tend to unphysical oszillations at internal and boundary layers. Therefore, additional stabilization is needed. A first linear stabilized surface finite element method on unfitted meshes is presented in [4]. The problem is stabilized using a standard SUPG approach. Estimates for the geometric error and the L^2 und H^1 error are established.

We consider a linear finite element method with local projection stabilization on fitted meshes. The idea of stabilizing by local projection was first introduced for the Stokes problem in [1] and by now is well established for diffusion-convection equations. On surfaces new challenges has to be handeled. Due to the non smooth approximation of a curved surface with triangles, geometric errors occur. Besides, partial integration on piecewise smooth surfaces leads to integrals over element boundaries.

We introduce a uniquely solvable stabilized problem formulation and present error estimations for the corresponding finite element method. Numerical experiments show the stabilizing properties

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An arbitrary Lagrangian-Eulerian finite element approach for free and moving boundary problems with large deformations

S. Basting
* Department of Mathematics, LS III, Dortmund University of Technology, Germany
steffen.basting@math.tu-dortmund.de

Classical moving mesh methods for free and moving boundary problems usually suffer from mesh distortion if large displacements of the problem domain or interfaces are considered.

In this talk, a finite element framework based on moving meshes is presented which aims at resolving these problems. The approach is based on the classical arbitrary Lagrangian-Eulerian (ALE) formulation of the fluid equations, but allows for temporal discontinuities of the ALE parametrization. In contrast to traditional ALE based mesh moving/front tracking methods, the deformation of the computational mesh is not directly derived from an extension of the interface displacement. Instead, the mesh is obtained from a variational mesh optimization technique which yields meshes that are aligned with the interface, retain connectivity and can be shown to be of optimal quality.

The presented approach allows for large deformations of the moving interface while preserving attractive features of front tracking methods: an accurate description of the interface, and the possibility of designing problem-tailored finite element spaces. The methodology is introduced and evaluated in the context of two-phase flows with surface tension, particulate flows and fluid-structure interaction problems.

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A Local Discontinuous Galerkin Method for the Navier-Stokes-Korteweg Equations

J.J.W. van der Vegt[1], Lulu Tian[1], Yan Xu[2] and J.G.M. Kuerten[1,3]

[1]University of Twente, P.O. Box 217, 7500 AE, Enschede, The Netherlands
{j.j.w.vandervegt,l.tian}@utwente.nl
[2]School of Mathematical Sciences, University of Science and Technology of China, Hefei, P.R. China
yxu@ustc.edu.cn [3]Eindhoven University of Technology, P.O. Box 513, 5600 MB, Eindhoven, The Netherlands
J.G.M.Kuerten@tue.nl

Diffuse interface methods provide an interesting way to compute multiphase flows since only a single set of equations is used. There is no need to explicitly compute the interface between the phases as in Volume of Fluid or Level Set methods. In this presentation we will consider the Navier-Stokes-Korteweg (NSK) equations to compute phase boundaries between liquid and vapour in compressible fluids. These equations contain, next to the viscous stress tensor, also the Korteweg tensor, and are suitable for a diffuse interface method. Both the isothermal and non-isothermal NSK equations will be discussed. The NSK equations are closed using the Van der Waals equation of state and phase transitions between liquid and vapour are distinguished using different values of the density. The addition of the Korteweg tensor to the Navier-Stokes equations results in a system of third order partial differential equations. In this presentation we will present a local discontinuous Galerkin discretisation for the NSK equations. An important feature of the LDG discretisation that we will present is that it uses the conservative form of the NSK equations. The LDG discretisation is therefore conservative and also satisfies the energy decay relation. Our approach is also suitable for the non-isothermal NSK equations. In contrast, many existing schemes for the isothermal NSK equations can not be easily extended to the non-isothermal case, since they require that the pressure only depends on the density, which is not the case for the non-isothermal NSK equations. Since the LDG discretisation of the NSK equations results in a stiff system of ordinary differential equations, we use a Singly Diagonally Implicit Runge-Kutta method for the time integration. This removes the rather stringent time-step restriction at a reasonable computational cost. The resulting algebraic equations from the SDIRK method are solved with a Newton method in combination with a GMRES Krylov solver with an incomplete LU decomposition as preconditioner. The LDG discretisation for the NSK equations will be demonstrated on a number of test cases, in particular simulations of the coalescence of two bubbles and the interaction of a bubble with a solid wall. Both isothermal and non-isothermal cases will be considered.

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Stabilized Finite Elements in the Context of Blood Damage Prediction

M. Behr* and L. Pauli
* Chair for Computational Analysis of Technical Systems, RWTH Aachen Unversity, Germany
{behr,pauli}@cats.rwth-aachen.de

Convection-diffusion-reaction (CDR) equations are used in many applications to model physical, chemical or physiological phenomena. In particular, macroscopic models of blood damage, be it hemolysis affecting red blood cells or thrombosis affecting platelets, are often employing systems of such CDR equations. These models are crucial for the prediction of biocompatibility in blood-handling devices, such as implantable blood pumps, oxygenators, and many others [1]. And while many stable and accurate discretization methods exist for the standard advection-diffusion equation, there is still need to develop robust methods for more complex types of CDR equations.

We focus on several alternatives to stabilize a general non-linear CDR of the type that arises in hemolysis modeling. The SUPG, GLS and multiscale approaches are compared, and the choice of the stabilization parameter is discussed. The role of discontinuity capturing is also discussed, and a comparison with a formerly used least-squares finite elements is included. The selected stabilized method is then used in realistic examples designed to test and calibrate the hemolysis model in a generic artificial blood pump of centrifugal type [2].

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Gradient-based nodal limiters for continuous finite elements

D. Kuzmin* and J. N. Shadid
* Institute of Applied Mathematics (LS III), TU Dortmund University
Vogelpothsweg 87, D-44227 Dortmund, Germany
kuzmin@math.uni-dortmund.de
Computational Mathematics Department, Sandia National Laboratories
P.O. Box 5800, Albuquerque, NM 87185, USA
jnshadi@sandia.gov

We present new linearity-preserving nodal limiters for enforcing discrete maximum principles in continuous (linear or bilinear) finite element approximations to transport problems with steep fronts. In the process of algebraic flux correction [1, 2, 3], the oscillatory antidiffusive part of a high-order base discretization is decomposed into internodal fluxes and constrained to be local extremum diminishing. The proposed nodal limiter functions are designed to be continuous and satisfy the principle of linearity preservation which implies the preservation of second-order accuracy in smooth regions. The use of limited nodal gradients makes it possible to circumvent angle conditions and guarantee that the discrete maximum principle holds on arbitrary meshes. A numerical study is performed for linear convection and anisotropic diffusion problems on uniform and distorted meshes in 2D.

References

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A Review of Variational Multiscale Methods for the Simulation of Turbulent Incompressible Flows

Volker John*

* Weierstrass Institute for Applied Analysis and Stochastics,
Leibniz Institute in Forschungsverbund Berlin e. V. (WIAS),
Mohrenstr. 39, 10117 Berlin, Germany
and
Free University of Berlin,
Department of Mathematics and Computer Science,
Arnimallee 6, 14195 Berlin, Germany

volker.john@wias-berlin.de

Various realizations of variational multiscale (VMS) methods for simulating turbulent incompressible flows have been proposed in the past fifteen years. All of these realizations obey the basic principles of VMS methods: They are based on the variational formulation of the incompressible Navier-Stokes equations and the scale separation is defined by projections. However, apart from these common basic features, the various VMS methods look quite different. In this review, the derivation of the different VMS methods is presented in some detail and their relation among each other and also to other discretizations is discussed. Another emphasis consists in giving an overview about known results from the numerical analysis of the VMS methods.

This is joint work with Naveed Ahmed, Tomás Chacón Rebollo, and Samuele Rubino

References

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Reduced basis method for turbulent flows

CHACÓN REBOLLO, T., DELGADO-ÁVILA, E.* AND GÓMEZ-MÁMOL, M.

Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, (SPAIN)

chacon@us.es, edelgado1@us.es, macarena@us.es

In this work we present a reduced basis model for the Smagorinsky turbulence model, as a first step to the construction of a reduced projection-based VMS turbulence model, which has a similar structure. This turbulence model includes a non-linear eddy diffusion term that we have to treat in order to solve efficiently our reduced basis model. We approximate this non-linear term using the Empirical Interpolation Method (*cf.* [2]), in order to obtain a linearised decomposition of the reduced basis Smagorinsky model.
This model is bases upon an *a posteriori* error estimation for Smagorinsky turbulence model. The theoretical development of the *a posteriori* error estimation is based on [4] and [5], according to the Brezzi-Rappaz-Raviart stability theory, and adapted for the non-linear eddy diffusion term.
The reduced basis Smagorinsky turbulence model is decoupled in a Online/Offline procedure. First, in the Offline stage, we construct hierarchical bases in each iteration of the Greedy algorithm, by selecting the snapshots which have the maximum *a posteriori* error estimation value. To assure the Brezzi inf-sup condition on our Reduced Basis space, we have to define a *supremizer* operator on the pressure solution, and enrich the reduced velocity space. Then, in the Online stage, we are able to compute a speedup solution of our problem, with a good accuracy.
Finally we present some numerical tests, programmed in FreeFem++ (*cf.* [1]), in which we show the speedup the computation of a solution of a steady flow in a backward-facing step.

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A discontinuous Galerkin solver for incompressible turbulent flow

B. Krank* and M. Kronbichler and W. A. Wall
Institute for Computational Mechanics, Technische Universität München, Germany
{krank,kronbichler,wall}@lnm.mw.tum.de

We present a solver using high-order discontinuous Galerkin discretizations for the incompressible Navier–Stokes equations in the moderate and high Reynolds number regime. This development is part of our efforts towards high-order discontinuous Galerkin methods for incompressible flow and interface problems. The incompressible solver with best performance is a DG method based on a velocity-correction time integration method that splits a time step into the explicit integration of the convective term, a pressure Poisson equation, and a Helmholtz-like equation for the viscous term. First and second derivatives are approximated by local Lax–Friedrichs fluxes and the interior penalty method, respectively. To stabilize the scheme in the underresolved (high Reynolds number) regime, the method is augmented by a grad-div stabilization traditionally applied in continuous finite elements. In the context of turbulent flow, our method targets implicit large-eddy simulation (LES), i.e., with implicit filtering and the assumption that the stabilization terms are an adequate representation of arising subgrid terms. We will present validation results of our implementation for several test cases including turbulent channel flow. In the presentation we will also compare the performance of this code with implicit LES of a continuous equal-order approach including the standard stabilization terms SUPG, PSPG and grad-div.

A stabilized finite element method for the two-field and three-field Stokes eigenvalue problems

D. Boffi, R. Codina* and Ö. Türk
Università di Pavia, Pavia, Italy
daniele.boffi@unipv.it
* Universitat Politècnica de Catalunya, Barcelona, Spain
ramon.codina@upc.edu
Gebze Technical University, Gebze/Kocaeli, Turkey
onder.turk@yandex.com

In this work the stabilized finite element approximation of the Stokes eigenvalue problems is considered for both the two-field (displacement-pressure) and the three-field (stress-displacement-pressure) formulations. The method presented is based on a subgrid scale concept, and depends on the approximation of the unresolvable scales of the continuous solution. In general, these techniques consist in the addition of a residual based term to the basic Galerkin formulation. Naturally, the application of a standard residual based stabilization method to a linear eigenvalue problem, leads to a quadratic eigenvalue problem in discrete form which is physically inconvenient. As a distinguished feature of the present study, we take the space of the unresolved subscales orthogonal to the finite element space, which promises a remedy to the above mentioned complication. In essence, we put forward that only if orthogonal projection is used, the residual is simplified and the use of term by term stabilization is allowed. Thus, we do not need to put the whole residual in the formulation, and the linear eigenproblem form is recovered properly. We prove that the method applied is convergent, and present the error estimates for the eigenvalues and the eigenfunctions. We report several numerical tests in order to illustrate that the theoretical results are validated.

Numercial studies regarding accuracy, robustness and solver efficiency of special "divergence-free" finite element discretizations

S. Turek*
* Institute for Applied Mathematics (LS III), TU Dortmund, Germany
ture@featflow.de

We discuss several, well-known as well as new, mixed finite element approaches for solving incompressible flow problems, where the exact divergence-free constraint for the velocity is relaxed via different discretely divergence-free constraints. We explain the numerical features of the proposed techniques and illustrate their characteristics w.r.t. accuracy and (multigrid) solver efficiency via prototypical flow problems of benchmarking character, particularly in the case of multiphase flow problems with significant surface tension.

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Towards pressure-robust mixed methods for the incompressible Navier–Stokes equations

A. Linke* and C. Merdon

* Weierstrass Institute, Germany
alexander.linke@wias-berlin.de

Weierstrass Institute, Germany
christian.merdon@wias-berlin.de

For more than thirty years it was thought that the construction of *pressure-robust* mixed methods for the incompressible Navier–Stokes equations, whose velocity error is pressure-independent, was practically impossible. However, a novel, quite universal construction approach shows that it is indeed rather easy to construct pressure-robust mixed methods. The approach repairs a certain L^2-orthogonality between gradient fields and discretely divergence-free test functions, and works for families of arbitrary-order mixed finite element methods, arbitrary-order discontinuous Galerkin methods, and finite volume methods. Novel benchmarks for the incompressible Navier–Stokes equations show that the approach promises significant speedups in computational practice compared to pure Galerkin discretizations or grad-div stabilization, whenever the continuous pressure is complicated.

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Pressure-robust 1st and 2nd-order finite element methods for Navier–Stokes discretisations

C. Merdon* and A. Linke
* Weierstrass Institute, Germany
christian.merdon@wias-berlin.de
Weierstrass Institute, Germany
alexander.linke@wias-berlin.de

Standard mixed finite element methods suffer from a pressure-dependence in the a priori velocity error estimates due to a violation of the L^2 orthogonality between (discretely) divergence-free test functions and gradients. This instability scales with the Reynolds number and can lead to severe velocity errors [4] which can be remedied by a variational crime in the spirit of [1,2,3] that employs local reconstruction of the test functions onto Raviart-Thomas or Brezzi-Douglas-Marini functions.

These local reconstruction operators are available for finite element methods of arbitrary order [3]. However, this talk focusses on efficient and minimal-invasive realisations for the first order Bernardi-Raugel and second order bubble enriched finite element methods and the observation, that in constrast to [3] only the bubble test functions have to be reconstructed. Several numerical examples involving transient and Navier-Stokes flows illustrate the robustness and potentials of the modified methods.

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Optimal L^2 Error for a Modified Crouzeix-Raviart Stokes Element

A. Linke and C. Merdon and W. Wollner*

Weierstrass Institute, Mohrenstr. 39, 10117 Berlin
[alexander.linke|Christian.Merdon]@wias-berlin.de

* Technische Universitt Darmstadt Fachbereich Mathematik Dolivostr. 15 64293 Darmstadt
wollner@mathematik.tu-darmstadt.de

The talk is concerned with optimal L^2 error estimates as presented in [1] for the velocity approximation in a nonconforming finite element approximation for the incompressible Stokes equation as proposed in [2].

Instead of the standard weak form, the method proposed in [2] modifies the right hand side of the Stokes problem, by considering the, lowest order, Raviart-Thomas projection of the test-function in the right hand side. This modification allows an estimation of the H^1-error of the velocity of optimal order depending on higher norms of the continuous velocity only – and not on the pressure.

The contribution of this presentation is to show that also optimal velocity estimates in L^2, independent of the pressure, can be derived. These estimates are complicated by the fact, that the lowest-order elements can not be handled by techniques useful for all higher-order elements – since the available polynomial degree does not allow for an $O(h^2)$ interpolation error in the L^2 norm to deal with the non-conformity error.

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An a priori error estimate of FE schemes for Navier-Stokes solutions with outflow condition

D. Arndt+, M. Braack* and G. Lube+

+University of Göttingen, Germany
{d.arndt,lube}@math.uni-goettingen.de

*Mathematical Seminar, University of Kiel, Germany
braack@math.uni-kiel.de

Artificial boundaries of computational domains for the Navier-Stokes equations should allow for simultaneous in- and outflow. A frequently used condition is the so called do-nothing condition [2] which arises automatically in the cases of variational formulations due to partial integration of the full stress tensor. Although this boundary condition may be useful in certain configurations, it is well-known that this boundary condition exhibit inherent instabilities and does not even ensure existence of weak solutions. Therefore, development and analysis of modifications of this 'outflow' condition are very important. This talk addresses recent results for the do-nothing condition [3, 4] for incompressible flows. We address in particular an a priori error estimate which extends the results of [1]. Moreover, we report on some numerical examples.

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On the construction of L_2 orthogonal elements of arbitrary order for Local Projection Stabilization

F. Schieweck, P. Skrzypacz* and L. Tobiska

* Nazarbayev University, School of Science and Technology, Kazakhstan
piotr.skrzypacz@nu.edu.kz

Otto von Guericke University of Magdeburg, Department of Mathematics, Germany
schiewec@ovgu.de, tobiska@ovgu.de

Based on the results from [1], we construct L_2 orthogonal conforming elements of arbitrary order for the Local Projection Scheme (LPS). L^2 orthogonal basis functions lead to a diagonal mass matrix which is an undisputable advantage for time discretizations. We prove that the constructed family of finite elements satisfies a local inf-sup condition, cf. [2]. Additionally, we determine the size of the local inf-sup constant in terms of the polynomial degree. Our numerical tests show that the discrete solution is oscillation-free and of optimal accuracy in the regions away from the boundary layer.

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Higher order variational time discretisations for the Oseen equations

G. Matthies*

* Technische Universität Dresden, Institut für Numerische Mathematik, 01062 Dresden, Germany

gunar.matthies@tu-dresden.de

We discuss different time discretisations of variational type applied to the time-dependent Oseen equations. As spatial discretisation, both inf-sup stable and equal-order pairs of finite element spaces for approximating velocity and pressure are considered.

Since Oseen problems are generally convection-dominated, a spatial stabilisation is needed. We will concentrate on local projection stabilisation methods which allow to stabilise the streamline derivative, the divergence constraint and, if needed, the pressure gradient separately.

To discretize in time, continuous Galerkin-Petrov methods (cGP) and discontinuous Galerkin methods (dG) as higher order variational time discretisation schemes are applied. These methods are known to be A-stable (cGP) or even strongly A-stable (dG). An adaption of the time postprocessing from [1] leads to numerical solutions which show for both velocity and pressure at the discrete time points a convergence rate of $2k + 1$ for dG(k) and $2k$ for cGP(k), respectively.

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Error analysis for a post-processed dG-solution in time of the transient Stokes problem

Shafqat Hussain and Friedhelm Schieweck*
Department of Mathematics, Capital University of Science and Technology Islamabad, Pakistan
shafqat.hussain@cust.edu.pk

* Institute for Analysis and Numerics, Otto-von-Guericke University Magdeburg, Germany
schiewec@ovgu.de

We study the discontinuous Galerkin time discretization (dG(k)-method) for the transient Stokes problem [3, 2] which is discretized in space by means of an inf-sup stable pair of finite element spaces (V_h, Q_h) for velocity and pressure, respectively. Here, the fully discrete solution $(u_h(t), p_h(t))$ on each time interval is a polynomial in time of order k with values in the finite element product space $V_h \times Q_h$. By means of a simple post-processing step proposed in [4], we can compute in a very inexpensive way a lifted solution $(\tilde{u}_h(t), \tilde{p}_h(t))$ which is globally continuous in time and a polynomial of order $k+1$ on each time interval. For this approximation $(\tilde{u}_h(t), \tilde{p}_h(t))$, we prove an optimal estimate for the velocity error in $L^2(L^2)$ of the higher order in time $\tau^{k+2} + h^{r+1}$, where τ denotes the time step size, h the mesh size and r the polynomial degree for the velocity approximation in V_h. Moreover, we prove an optimal $L^2(L^2)$ estimate for the pressure error of the order $\tau^{k+2} + h^r$, where the polynomial degree for the pressure approximation in Q_h is $r-1$ due to the inf-sup condition. Key ingredients of the analysis are a special higher order interpolate in time of the exact solution and a special stability estimate for the lifted velocity error (for both see [1]) applied in the discretely divergence free subspace of V_h as well as the proof of superconvergence of the error in the time derivative for the velocity.

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Participants:

1. **Daniel Arndt**, Dr., University of Heidelberg, IWR, Germany
daniel.arndt@iwr.uni-heidelberg.de

2. **Steffen Basting**, Dipl. Technomath., Department of Mathematics, LS III, Dortmund University of Technology, Germany
steffen.basting@math.tu-dortmund.de

3. **Marek Behr**, Prof. Ph.D., RWTH Aachen University, Germany
behr@cats.rwth-aachen.de

4. **Malte Braack**, Prof. Dr., University of Kiel, Germany
braack@math.uni-kiel.de

5. **Ramon Codina**, Prof. Dr., Universitat Politècnica de Catalunya, Barcelona, Spain
ramon.codina@upc.edu

6. **Enrique Delgado-Ávila**, PhD student in Mathematics, University of Seville, Spain
edelgado1@us.es

7. **Andreas Hahn**, Dipl.-Math., Otto-von-Guericke University Magdeburg, Germany
Andreas.Hahn@ovgu.de

8. **Volker John**, Prof. Dr., Weierstrass Institute, Germany
john@wias-berlin.de

9. **Utku Kaya**, M.Sc., University of Kiel, Germany
kaya@math.uni-kiel.de

10. **Benjamin Krank**, Dipl.-Ing., Institute for Computational Mechanics, Technical University Munich, Germany
krank@lnm.mw.tum.de

11. **Dmitri Kuzmin**, Prof. Dr., Department of Mathematics, LS III, Dortmund University of Technology, Germany
kuzmin@math.uni-dortmund.de

12. **Alexander Linke**, Dr., Weierstrass Institute, Germany
alexander.linke@wias-berlin.de

13. **Gert Lube**, Prof. Dr., Georg-August University Göttingen, NAM, Germany
lube@math.uni-goettingen.de

14. **Gunar Matthies**, Prof. Dr., Technical University Dresden, Germany
gunar.matthies@tu-dresden.de

15. **Christian Merdon**, Dr. rer. nat., Weierstrass Institute Berlin, Germany
Christian.Merdon@wias-berlin.de

16. **Friedhelm Schieweck**, apl. Prof. Dr., Otto-von-Guericke University Magdeburg, Germany
schiewec@ovgu.de

17. **Philipp W. Schroeder**, M.Sc. , Georg-August-University Göttingen, Institute for Numerical and Applied Mathematics, Germany
philippwilliam@aol.com

18. **Kristin Simon**, Dipl.-Math., Otto-von-Guericke University Magdeburg, Germany
Kristin.Simon@ovgu.de

19. **Piotr Skrzypacz**, Dr. (Assistant Prof.), Nazarbayev University, School of Science and Technology, Kazakhstan
piotr.skrzypacz@nu.edu.kz

20. **Lutz Tobiska**, Prof. Dr., Otto-von-Guericke University Magdeburg, Germany
tobiska@ovgu.de

21. **Stefan Turek**, Prof. Dr., Department of Mathematics, LS III, Dortmund University of Technology, Germany
stefan.turek@math.tu-dortmund.de

22. **Jaap van der Vegt**, Prof., Department of Applied Mathematics, University of Twente, Enschede, P.O. Box 217, 7500 AE, The Netherlands
j.j.w.vandervegt@utwente.nl

23. **Winnifried Wollner**, Prof. Dr., Technical University Darmstadt, Germany
wollner@mathematik.tu-darmstadt.de